JHEP HYPER VERSION

Received: *August 16, 2001*, Accepted: *November 18, 2001*

Visible branes with negative tension in heterotic M-theory

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Abstract: It is shown that there exist large classes of BPS vacua in heterotic M-theory which have negative tension on the visible orbifold plane, positive tension on the hidden plane and positive tension, physical five-branes in the bulk space. Explicit examples of such vacua are presented. Furthermore, it is demonstrated that the ratio, $\beta/|\alpha|$, of the bulk five-brane tension to the visible plane tension can, for several large classes of such vacua, be made arbitrarily small. Hence, it is straightforward to find vacua with the properties required in the examples of the Ekpyrotic theory of cosmology — a visible brane with negative tension and $\beta/|\alpha|$ small. This contradicts recent claims in the literature.

Keywords: M-Theory, Physics of the Early Universe.

Contents

1. Introduction

In a recent paper [1], a new theory of the early universe, called the Ekpyrotic Universe, was introduced. This theory attempts to resolve the cosmological horizon, flatness and monopoles problems and to generate a nearly scale-invariant spectrum of energy density perturbations without requiring any period of inflation. Although, in principle, Ekpyrotic theory applies to any brane world scenario [2]–[8], in [1] it was specifically demonstrated in two concrete examples, AdS spaces [7] and heterotic M-theory [2, 3, 9]. In both cases, it was assumed that the tension of the visible brane, that is, the brane supporting our observable universe, is negative, whereas the tension of the hidden brane, that is, the brane communicating with our world only through the extra dimension, is positive. Although the Ekpyrotic scenario does not, necessarily, require such an assumption, it was helpful in constructing the theories presented in [1]. It was further assumed, fundamentally, that such theories contain physical branes in the bulk space. In heterotic M-theory, where such questions can be analyzed in detail, it is clear that these assumptions are valid for a wide class of physically relevant vacua. In this paper, we will discuss and explicitly construct large classes of BPS vacua in heterotic M-theory that have negative tension visible branes, positive tension hidden branes and physical, positive tension five-branes in the bulk space. These five-branes are wrapped on holomorphic curves in the Calabi-Yau three-fold background.

In Ekpyrotic theory, the ratio of the bulk five-brane tension, β, to the magnitude of the tension, $|\alpha|$, of the visible brane arises as an important parameter. Although the Ekpyrotic scenario is potentially applicable for any value of this ratio, the theory is easiest to analyze when $\beta/|\alpha|$ is small. In [1], a sample value of this ratio was chosen to be of order 10^{-4}. In the same paper, it was demonstrated in a second example that values of $\beta/|\alpha|$ as large as $1/10$ are also acceptable. Be this as it may, it is useful to demonstrate that small values of this ratio can be obtained in brane world scenarios, so that we understand how much flexibility we have in designing models. In this paper, we show explicitly that arbitrarily small values of the ratio $\beta/|\alpha|$ can be obtained in physically relevant BPS vacua of heterotic M-theory. These vacua have negative tension visible branes, positive tension hidden branes and bulk space five-branes with positive tension.

Specifically, we do the following. In section 2, we present the relevant structure of effective five-dimensional heterotic M-theory without bulk five-branes. This is extended to include bulk five-branes in section 3. We define the tension of each orbifold boundary plane and bulk five-brane in terms of characteristic classes. We demonstrate that, for generic stable holomorphic vector bundles associated with the orbifold fixed planes, the tension of the visible plane can be negative while the tension of the hidden plane is positive. The tension of the bulk brane must always be positive since it corresponds to an effective class, that is, it is wrapped on a holomorphic curve of positive volume. In section 4, we review the formalism for computing the second Chern class of the tangent bundle of elliptically fibered Calabi-Yau three-folds with either del Pezzo or Hirzebruch base surfaces, and the Chern classes for stable, holomorphic vector bundles over such manifolds. This formalism is applied in section 5, where we present two explicit heterotic M-theory BPS vacua which manifestly exhibit negative tension on the visible plane and positive tension on the hidden plane and bulk branes. In both of these examples, the visible sector supports an SU(5) grand unified theory with three families of quarks and leptons and the hidden sector an E_7 gauge group. In section 6, by discussing the relative volume of effective curves, we show that there is a large class of M-theory BPS vacua whose branes exhibit the desired tensions, that is, negative and positive tensions on the visible and hidden branes respectively, but not manifestly. We present two explicit vacua of this type. The first has the same gauge group and matter structure as previously. The second example has an SU(5) grand unified theory with three families of quarks and leptons on the visible plane with an E_6 gauge group on the hidden plane. In section 7, we compute the ratio, $\beta/|\alpha|$, of the bulk brane tension to the magnitude of the visible plane tension for each of the four explicit M-theory vacua presented in this paper. While for two of these vacua this ratio is of order unity, we show that for the remaining two classes of vacua, this ratio can be made arbitrarily small. In the conclusion, section 8, we argue that our results can be extended to other elliptically fibered Calabi-Yau three-folds, as well as other choices of gauge groups. Finally, in an appendix, we present a proof that the volume of effective curves associated with the base can be made arbitrarily large with respect to the volume of the elliptic fiber. This fact is essential in the discussions in sections 6 and 7.

In a recent paper by Kallosh, Kofman and Linde [10], it was suggested that it is not possible to have negative tension visible branes and positive tension hidden branes in heterotic M-theory. This is incorrect. We show in section 3 of this paper that it is clear, by construction, that heterotic M-theory vacua with non-standard gauge embeddings (that is, general stable, holomorphic vector bundles) can have the tensions as stated in [1]. Furthermore, it is very simple to create large classes of such vacua, as is done in sections 5 and 6 of this paper. The authors of [10] refer primarily to the so-called standard embeddings. These embeddings, indeed, have the tensions reversed. However, these vacua are long since known not to allow five-branes in the bulk space and, hence, are clearly not appropriate for the Ekpyrotic scenario [1]. In their revised version, the authors of [10] claim that even for non-standard embeddings it is not possible to get a negative tension visible brane and a positive tension hidden brane. They state as the reason that the gauge group on the visible sector must be smaller than the gauge group on the hidden sector. They conclude from this that the tension on the visible brane, therefore, must be larger than the tension of the hidden brane. This also is not correct, as all the examples in this paper will clearly demonstrate. Quite to the contrary, we show that it is straightforward to construct vacua in heterotic M-theory satisfying the requirements of examples of Ekpyrotic cosmology presented in ref. [1].

2. Effective theory without bulk five-branes

As first shown in [9], cancellation of both gravitational and gauge anomalies on the orbifold fixed planes requires that the Bianchi identity associated with the three-form C_{IJK}, $I, J = 0, \ldots, 9, 11$, be modified to

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = -4\sqrt{2}\pi \left(\frac{\kappa}{4\pi}\right)^{2/3} (J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11} - \pi\rho))_{\bar{I}\bar{J}\bar{K}\bar{L}}\,, \tag{2.1}$$

where $\bar{I}, \bar{J}, \ldots = 0, \ldots, 9$ are ten-dimensional indices and $G = dC$ is the field strength. The sources are defined by

$$J^{(n)} = c_2(V^n) - \frac{1}{2}c_2(TX) \qquad n = 1, 2\,, \tag{2.2}$$

and

$$c_2(V^n) = \frac{1}{16\pi^2} trF^n \wedge F^n\,, \qquad c_2(TX) = \frac{1}{16\pi^2}\,\mathrm{tr}\,R \wedge R\,, \tag{2.3}$$

where F^n is the field strength associated with the gauge theory on the n-th plane, and R is the Ricci tensor of the Calabi-Yau manifold. Note that $c_2(V^n)$ and $c_2(TX)$ are the second Chern class of the vector bundle on the n-th boundary plane and the second Chern class of the Calabi-Yau tangent bundle respectively. Integrating (2.1) over a five-cycle which spans the orbifold interval and is otherwise an arbitrary four-cycle in the Calabi-Yau three-fold, we find

$$c_2(V^1) + c_2(V^2) - c_2(TX) = 0\,. \tag{2.4}$$

It follows that the sources must satisfy

$$J^{(1)} = -J^{(2)}\,. \tag{2.5}$$

Note that this implies that the bundles on the two fixed planes must, in general, be different.

The effective five-dimensional theory with two four-dimensional boundaries, but without bulk five-branes, was computed in [3] and was found to be

$$S_5 = S_{\text{bulk}} + S_{\text{boundary}}\,, \tag{2.6}$$

where

$$S_{\text{bulk}} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\left(R + \frac{1}{2}V^{-2}\partial_\alpha V\partial^\alpha V + \cdots\right) \tag{2.7}$$

and

$$S_{\text{boundary}} = -\frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(1)}}\sqrt{-g}\,V^{-1}\alpha_i^{(1)}b^i - \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(2)}}\sqrt{-g}\,V^{-1}\alpha_i^{(2)}b^i\,. \tag{2.8}$$

Note the we have explicitly written only those terms in (2.7) and (2.8) that are relevant to this discussion. The various quantities that make up this effective theory are defined as follows. First,

$$\kappa_5^2 = \frac{\kappa^2}{v}\,, \tag{2.9}$$

where κ is the 11-dimensional gravitational coupling constant and v is a 6-dimensional reference volume. The moduli fields arise from the structure of the 11-dimensional line element, which has the form

$$ds^2 = V^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dx^A dx^B\,. \tag{2.10}$$

The indices take values $\alpha, \beta = 0, 1, 2, 3, 11$ and $A, B = 4, \ldots, 9$ with the latter corresponding to the six real coordinates of the Calabi-Yau three-fold. Then

$$V = \frac{1}{v}\int_X \sqrt{^6 g}\,, \tag{2.11}$$

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where ${}^6g$ is the determinant of g_{AB}. Now define metric

$$\Omega_{AB} = V^{-1/3} g_{AB} \tag{2.12}$$

and let $\omega_{a\bar{b}} = i\Omega_{a\bar{b}}$ be its Kähler form, where a, b are the complex coordinates corresponding to A, B. Choosing a basis $\omega_{iAB}, i = 1, \ldots, h^{1,1}$ of harmonic (1,1)-forms of $H^{(1,1)}(X)$, whose Poincaré duals, $\mathcal{C}_i$, are cycles in $H_4(X, \mathbb{Z})$, we can expand

$$\omega_{AB} = b^i \omega_{iAB}\,, \tag{2.13}$$

which defines the $h^{1,1}$ moduli b^i. Finally, the coefficients $\alpha_i^{(n)}$ are given by

$$\alpha_i^{(n)} = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_i} J^{(n)}\,. \tag{2.14}$$

The $h^{1,1} = h_4$ four-cycles $\mathcal{C}_i$ span $H_4(X, \mathbb{Z})$ and, by definition, satisfy

$$\int_{\mathcal{C}_i} \nu = \int_X \omega_i \wedge \nu \tag{2.15}$$

for any four-form ν. Note from equations (2.5) and (2.14) that

$$\alpha_i^{(1)} = -\alpha_i^{(2)} \tag{2.16}$$

for any value of i.

For arbitrary four-form ν,

$$b^i \int_{\mathcal{C}_i} \nu = \int_{b^i \mathcal{C}_i} \nu = \int_X b^i \omega_i \wedge \nu\,. \tag{2.17}$$

It then follows from (2.13) and (2.15) that

$$b^i \int_{\mathcal{C}_i} \nu = \int_{\mathcal{C}_\omega} \nu\,, \tag{2.18}$$

where $\mathcal{C}_\omega$ is the Poincaré dual four-cycle to the Kähler class ω. We conclude that

$$\alpha_i^{(n)} b^i = \alpha^{(n)}\,, \tag{2.19}$$

where

$$\alpha^{(n)} = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} J^{(n)}\,. \tag{2.20}$$

Again, one finds from (2.5) that

$$\alpha^{(1)} = -\alpha^{(2)}\,. \tag{2.21}$$

We conclude that the boundary part of the action is given by

$$S_{\text{boundary}} = -\frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(1)}} \sqrt{-g}\, V^{-1} \alpha^{(1)} - \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(2)}} \sqrt{-g}\, V^{-1} \alpha^{(2)}\,, \tag{2.22}$$

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where, using (2.2) and (2.20) we have

$$\alpha^{(n)} = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} \left(c_2(V^n) - \frac{1}{2} c_2(TX) \right) . \tag{2.23}$$

Note that, on the boundaries, the Kähler form ω and moduli b^i are generically functions of x^μ, $\mu = 0, 1, 2, 3$ and, hence, so are $\alpha^{(n)}$. However, in this paper, we are interested in the brane tensions associated with the universal BPS vacuum solutions discussed in [3]. In these solutions, the b^i moduli are constants. It follows that, for these vacua, the Kähler form and, therefore, $\alpha^{(n)}$ are constants. We will assume this for the remainder of this paper.

From the form of the action given by (2.6), (2.7) and (2.22), we see that the coefficient $\alpha^{(n)}$ defines the tension on the n-th orbifold plane. Specifically, if $\alpha^{(n)} > 0$ then the tension of the associated boundary plane is positive, whereas if $\alpha^{(n)} < 0$ the tension on that orbifold plane is negative. It is clear from the expression for $\alpha^{(n)}$ in (2.23) that its value and sign depend on the explicit choice of the Calabi-Yau three-fold (thus specifying $c_2(TX)$) and on the holomorphic vector bundle V^n (thus specifying $c_2(V^n)$).

The simplest example one can present is the so-called standard embedding, where one fixes a Calabi-Yau three-fold and chooses the two holomorphic vector bundles so that $V^1 = TX$ and $V^2 = 0$. It follows that

$$c_2(V^1) = c_2(TX) , \qquad c_2(V^2) = 0 . \tag{2.24}$$

Note that these Chern classes satisfy the topological condition given in (2.4). From (2.23) we find

$$\alpha^{(1)} = \frac{\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} c_2(TX) , \tag{2.25}$$

which is positive. Hence, the tension on the 1-plane is positive. Similarly, the coefficient $\alpha^{(2)}$ is negative, consistent with expression (2.21), and, hence, the tension on the 2-plane is negative. The standard embedding can be given a particle physics interpretation. Note that the bundle $V^1 = TX$ must have structure group $G = \mathrm{SU}(3)$, whereas bundle V^2 is zero. It follows that the surviving low-energy gauge symmetry on the 1-plane is $H = E_6$ (the commutant of SU(3) in E_8). The gauge symmetry on the 2-plane remains E_8. Since E_6 might be interpreted as a viable grand unification group, it is conventional to call the 1-plane the "visible" or "observable" sector and the 2-plane the "hidden" sector. With these definitions, we see that -in the standard embedding- the visible sector has positive tension and the hidden sector negative tension.

Although they are well-known, bundles satisfying the standard embedding conditions suffer from a number of difficulties. To begin with, as we have said, they lead to an E_6 grand unified theory (GUT) in the visible sector which, having a large rank and dimension, is more difficult to break to the standard model group $SU(3)_C \times SU(2)_L \times U(1)_Y$. In addition, although it can be done, it is hard to obtain a spectrum with three light families of quarks and leptons. Furthermore, even though the standard embedding is natural in the context of $E_8 \times E_8$ heterotic superstring theory, there is no reason for preferring it in M-theory, where all consistent stable, holomorphic vector bundle vacua are on an equal footing. Finally, as we will show explicitly in the next section, since the standard embedding satisfies the topological condition (2.4), it does not admit any five-branes in the five-dimensional bulk space. Therefore, the standard embedding cannot be used in the recently proposed Ekpyrotic cosmological theory [1], or, for that matter, in the variant given in [10].

For all these reasons, it is of interest to construct "non-standard embeddings", that is, vacua in which the spin connection of the Calabi-Yau three-fold is not embedded in the gauge group. For vacua without bulk five-branes, such non-standard embeddings are described by stable, holomorphic vector bundles satisfying the topological condition (2.4), but not (2.24). One property of these vacua is immediately apparent. By choosing vector bundles V^1 and V^2 so that

$$\int_{\mathcal{C}_\omega} c_2(V^1) < \frac{1}{2} \int_{\mathcal{C}_\omega} c_2(TX), \tag{2.26}$$

one can make $\alpha^{(1)} < 0$ and, using (2.21), $\alpha^{(2)} > 0$. As we will show by explicit example in the next section, vector bundles V^1 satisfying (2.26) can break E_8 in the visible sector down to small gauge groups, such as SU(5). The examples in this paper are easily extended to other small gauge groups, such as SO(10) and the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. It follows, therefore, that non-standard embeddings can easily produce vacua with negative tension on the visible orbifold plane and positive tension on the hidden sector fixed plane. It is not too difficult to produce examples of non-standard embeddings satisfying (2.4) and (2.26). However, such vacua continue to suffer from the last difficulty mentioned above, namely that, since they do not admit bulk five-branes, they cannot be used as vacua for the Ekpyrotic cosmological theory. We turn, therefore, to more generalized vacua of M-theory that do admit bulk five-branes.

3. Effective theory with bulk five-branes

When N bulk five-branes, located at coordinates x_i for $i = 1, \ldots, N$ in the 11-direction, are present in the vacuum, cancellation of their worldvolume anomalies, as well as the gravitational and gauge anomalies on the orbifold fixed planes, requires that Bianchi identity (2.1) be further modified to

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = -4\sqrt{2}\pi \left(\frac{\kappa}{4\pi}\right)^{2/3} (J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11} - \pi\rho) + \sum_{i=1}^{N} \hat{J}^{(i)}\delta(x^{11} - x_i))_{\bar{I}\bar{J}\bar{K}\bar{L}} \,. \tag{3.1}$$

Each five-brane source $\hat{J}^{(i)}$ is defined to be the four-form which is Poincaré dual to the holomorphic curve in the Calabi-Yau three-fold around which the i-th five-brane is wrapped. If we define the five-brane class

$$W = \sum_{i=1}^{N} \hat{J}^{(i)} \,, \tag{3.2}$$

then the topological condition (2.4) is modified to

$$c_2(V^1) + c_2(V^2) - c_2(TX) + W = 0 \,. \tag{3.3}$$

As we will see, the addition of bulk five-branes makes solution of this topological condition considerably simpler.

The effective five-dimensional theory with two four-dimensional boundaries and N bulk five-branes was discussed in [11]. For the remainder of this paper, we will assume, for simplicity, that all bulk five-branes are located at the same point in the orbifold direction. This assumption corresponds to choosing a specific region of the moduli space of W and is sufficient for our purposes. A detailed analysis of the full moduli space of five-brane classes W was presented in [12]. None of the conclusions arrived at in this section will change if one considers other regions of moduli space. With this simplifying assumption, the effective theory is found to be

$$S_5 = S_{\text{bulk}} + S_{\text{boundary}} + S_{\text{5-brane}} \,, \tag{3.4}$$

where the first few terms of S_{bulk} are given in (2.7). The boundary and five-brane actions are

$$S_{\text{boundary}} = -\frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(1)}} \sqrt{-g}\, V^{-1} \alpha^{(1)} - \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(2)}} \sqrt{-g}\, V^{-1} \alpha^{(2)} \,, \tag{3.5}$$

where

$$\alpha^{(n)} = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} \left(c_2(V^n) - \frac{1}{2} c_2(TX) \right) , \tag{3.6}$$

and

$$S_{\text{5-brane}} = -\frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(\text{5-brane})}} \sqrt{-g}\, V^{-1} \beta \,, \tag{3.7}$$

with

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} W \,. \tag{3.8}$$

Note from topological condition (3.3) that

$$\alpha^{(1)} + \alpha^{(2)} + \beta = 0 \,. \tag{3.9}$$

Before considering non-standard embeddings, let us briefly discuss how the standard embedding, that is, holomorphic vector bundles satisfying (2.24), fit into this context. For the standard embedding, it follows from (3.6) that $\alpha^{(1)} = -\alpha^{(2)}$ and, hence, from (3.9) that $\beta = 0$. That is, the standard embedding always has vanishing five-brane class W and, therefore, no five-branes in the bulk, as anticipated in the previous section. We now proceed to the construction of non-standard embedding vacua. To do this, we need to introduce specific Calabi-Yau three-folds and give a method for constructing stable, holomorphic vector bundles over them. Once this is accomplished, we can use expressions (3.6) and (3.8) to calculate the tension of the visible and hidden orbifold planes, as well as the bulk five-brane.

4. Vector bundles and Chern classes

In this paper, we will consider Calabi-Yau three-folds that are elliptically fibered over a base surface. The stable, holomorphic vector bundles over these Calabi-Yau three-folds will be constructed using the methods introduced in [13, 14, 15] and [16]–[20]. Here, we only state the results that we specifically need to formulate realistic particle physics models and to compute their brane tensions. We refer the reader to [16]–[19] where these topics are discussed in detail.

The second Chern class of an elliptically fibered Calabi-Yau three-fold X over a base surface B is given by

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)\,, \tag{4.1}$$

where $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of the base manifold respectively and σ is the zero section of X. B is restricted to be either a del Pezzo, Hirzebruch or Enriques surface, or a blow-up of a Hirzebruch surface. The Chern classes are known for all of these. In this paper we will consider, for specificity, del Pezzo and Hirzebruch surfaces only. The other allowed bases can also be considered. These have the following Chern classes. For del Pezzo surfaces dP_r, $r = 1, \ldots, 8$, the Chern classes are given by

$$c_1(dP_r) = 3l - \sum_{i=1}^{r} E_i\,, \qquad c_2 = 3 + r\,, \tag{4.2}$$

where l and E_i, $i = 1, \ldots, r$ are effective curves spanning $H_2(dP_r)$. Their intersection numbers are

$$l \cdot l = 1\,, \qquad l \cdot E_i = 0\,, \qquad E_i \cdot E_j = -\delta_{ij}\,. \tag{4.3}$$

For Hirzebruch surfaces F_r, r any non-negative integer, the Chern classes are given by

$$c_1(F_r) = 2S + (r+2)\mathcal{E}\,, \qquad c_2(F_r) = 4\,, \tag{4.4}$$

where S and $\mathcal{E}$ are effective curves spanning $H_2(F_r)$. Their intersection numbers are

$$S \cdot S = -r\,, \qquad S \cdot \mathcal{E} = 1\,, \qquad \mathcal{E} \cdot \mathcal{E} = 0\,. \tag{4.5}$$

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The Chern classes for a family of stable, holomorphic vector bundles V over elliptically fibered Calabi-Yau three-folds were computed in [13, 15]. For structure group $G = \mathrm{SU}(n)$, they were found to be

$$c_2(V) = \sigma\eta - \frac{1}{24}c_1(B)^2(n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\eta(\eta - nc_1(B)) \tag{4.6}$$

and

$$c_3(V) = 2\lambda\sigma\eta(\eta - nc_1(B)) \,. \tag{4.7}$$

Here η is any effective curve in the base surface B and λ is a rational number subject to the constraints

$$n \text{ is odd}\,, \qquad \lambda = m + \frac{1}{2} \tag{4.8}$$

or

$$n \text{ is even}\,, \qquad \lambda = m\,, \quad \eta = c_1(B) \bmod 2\,, \tag{4.9}$$

where m is an integer.

It follows from (3.3) that

$$W = c_2(TX) - c_2(V^1) - c_2(V^2)\,. \tag{4.10}$$

Using expressions (4.1) and (4.6), we can write the five-brane class W as

$$W = W_B + a_f F\,, \tag{4.11}$$

where

$$W_B = \sigma(12c_1(B) - \eta^{(1)} - \eta^{(2)}) \tag{4.12}$$

is the component of the class associated with the base B and

$$\begin{aligned} a_f = c_2(B) &+ \left(11 + \frac{n^{(1)3} - n^{(1)}}{24} + \frac{n^{(2)3} - n^{(2)}}{24}\right) c_1(B)^2 - \\ &- \frac{1}{2}n^{(1)}\left(\lambda^{(1)2} - \frac{1}{4}\right)\eta^{(1)}(\eta^{(1)} - n^{(1)}c_1(B)) - \\ &- \frac{1}{2}n^{(2)}\left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}(\eta^{(2)} - n^{(2)}c_1(B)) \end{aligned} \tag{4.13}$$

is the integer coefficient associated with the elliptic fiber class F. Here, the superscript (i), $i = 1, 2$ refers to the stable, holomorphic vector bundle V^i on the i-th orbifold plane. As discussed in [16], there is an important additional constraint that five-brane class W must satisfy. That is, in order for W to represent a set of physical five-branes in the bulk it must be an effective class. It was shown in [16] that this will be the case if and only if

$$W_B \text{ is effective in } B\,, \qquad a_f \geq 0\,. \tag{4.14}$$

This condition strongly constrains the allowed vacua. The results presented in this section allow one to construct realistic vacua of M-theory, to which we now turn.

5. Physical vacua and brane tension

There are many physically realistic M-theory vacua that will have the three properties that we wish to explore, namely, negative tension on the visible orbifold plane, positive tension on the hidden plane and non-vanishing five-branes in the bulk. To simplify our discussion, in this paper we will assume that V^1, the stable, holomorphic vector bundle on the 1-plane, has structure group

$$G^{(1)} = \mathrm{SU}(5)\,. \tag{5.1}$$

That is, we choose

$$n^{(1)} = 5\,. \tag{5.2}$$

It follows that the low energy gauge group on the 1-plane is

$$H^{(1)} = \mathrm{SU}(5)\,, \tag{5.3}$$

which is the commutant subgroup of $G^{(1)} = \mathrm{SU}(5)$ in E_8. Furthermore, it was shown in [20] that the number of quark and lepton families is given by

$$N_{\mathrm{family}} = \frac{c_3(V)}{2}\,. \tag{5.4}$$

We, henceforth, demand that the number of families on the 1-plane be the observed number, that is, three. It follows from (4.7) that

$$\lambda^{(1)}\sigma\eta^{(1)}\left(\eta^{(1)} - 5c_1(B)\right) = 3\,. \tag{5.5}$$

Conditions (5.2) and (5.5) imply that the 1-plane supports an SU(5) GUT with three families of quarks and leptons. That is, these conditions lead to a physically realistic particle model on the 1-plane and, hence, this plane is the visible sector. We find from (3.6), (4.1) and (4.6) that the tension of the visible sector is

$$\alpha^{(1)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}\left(c_2(V^1) - \frac{1}{2}c_2(TX)\right), \tag{5.6}$$

where

$$c_2(V^1) - \frac{1}{2}c_2(TX) = (\eta^{(1)} - 6c_1(B))\sigma - \frac{1}{2}\left(c_2(B) + 21c_1(B)^2 - \frac{15}{\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right)\right)F \tag{5.7}$$

and we have used (5.2) and (5.5). Note that since we have chosen $n^{(1)}$ to be odd, it follows from (4.8) that

$$\lambda^{(1)} = m^{(1)} + \frac{1}{2}\,, \tag{5.8}$$

where $m^{(1)}$ is any integer and $\eta^{(1)}$ must be effective in B but is otherwise unconstrained.

Similarly, to simplify the discussion, we will in this section assume that V^2, the stable, holomorphic vector bundle on the 2-plane, has structure group

$$G^{(2)} = \mathrm{SU}(2)\,. \tag{5.9}$$

That is, we choose

$$n^{(2)} = 2\,. \tag{5.10}$$

It follows that the low energy gauge group on the 2-plane is

$$H^{(2)} = E_7\,, \tag{5.11}$$

which is the commutant subgroup of $G^{(2)} = \mathrm{SU}(2)$ in E_8. We need impose no further restrictions. Condition (5.10) implies that the 2-plane supports an E_7 supergauge theory with a certain number of matter multiplets that we need not specify. That is, the 2-plane is typical of a hidden sector. We find from (3.6), (4.1) and (4.6) that the tension of the hidden sector is

$$\alpha^{(2)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}\left(c_2(V^2) - \frac{1}{2}c_2(TX)\right), \tag{5.12}$$

where

$$\begin{aligned} c_2(V^2) - \frac{1}{2}c_2(TX) = {} & \left(\eta^{(2)} - 6c_1(B)\right)\sigma - \frac{1}{2}\bigg(c_2(B) + \frac{23}{2}c_1(B)^2 - \\ & - 2\left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}(\eta^{(2)} - 2c_1(B))\bigg)F \end{aligned} \tag{5.13}$$

and we have used (5.10). Note from (4.9) that, since we have chosen $n^{(2)}$ to be even, we must have

$$\lambda^{(2)} = m^{(2)}\,, \tag{5.14}$$

where $m^{(2)}$ is any integer. Furthermore, in addition to being effective in B, $\eta^{(2)}$ must satisfy

$$\eta^{(2)} = c_1(B) \bmod 2\,. \tag{5.15}$$

Finally, using the conditions (5.2), (5.5) and (5.10), the expressions for the five-brane class given in (4.12) and (4.13) simplify to

$$W_B = \sigma(12c_1(B) - \eta^{(1)} - \eta^{(2)}) \tag{5.16}$$

and

$$\begin{aligned} a_f = c_2(B) + \left(\frac{65}{4}\right)c_1(B)^2 - \frac{15}{2\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right) \\ - \left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}(\eta^{(2)} - 2c_1(B))\,. \end{aligned} \tag{5.17}$$

In terms of these quantities, the tension on the five-brane is given by

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} W \,. \tag{5.18}$$

We now present several heterotic M-theory vacua within this context that explicitly have negative tension on the visible 1-plane, positive tension on the hidden 2-plane and physical five-branes in the bulk with positive brane tension.

Example 1: dP_8**.** In this example, we will choose the base surface of the elliptically fibered Calabi-Yau three-fold to be

$$B = dP_8 \,. \tag{5.19}$$

Using (4.2), (4.3) and the fact that $c_1(dP_r)^2 = 9 - r$, it follows that, for $r = 8$, eq. (5.7) becomes

$$c_2(V^1) - \frac{1}{2}c_2(TX) = \left(\eta^{(1)} - 6c_1(B)\right)\sigma - \frac{1}{2}\left(32 - \frac{15}{\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right)\right)F \,, \tag{5.20}$$

eq. (5.13) simplifies to

$$\begin{aligned} c_2(V^2) - \frac{1}{2}c_2(TX) = {} & \left(\eta^{(2)} - 6c_1(B)\right)\sigma - \\ & -\frac{1}{2}\left(\frac{45}{2} - 2\left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}\left(\eta^{(2)} - 2c_1(B)\right)\right)F \,, \end{aligned} \tag{5.21}$$

and the five-brane class specified by (5.16) and (5.17) becomes

$$W_B = \sigma\left(12c_1(B) - \eta^{(1)} - \eta^{(2)}\right) \tag{5.22}$$

and

$$a_f = \frac{109}{4} - \frac{15}{2\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right) - \left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}\left(\eta^{(2)} - 2c_1(B)\right) , \tag{5.23}$$

where

$$c_1(dP_8) = 3l - \sum_{i=1}^{8} E_i \,. \tag{5.24}$$

We now choose

$$\lambda^{(1)} = -\frac{1}{2} \,, \qquad \eta^{(1)} = 2c_1(dP_8) \,. \tag{5.25}$$

Note that this value of $\lambda^{(1)}$ satisfies (5.8) and that $\eta^{(1)}$ is effective since $c_1(dP_8)$ is. It is straightforward to show that these choices satisfy the three-family condition (5.5) on the visible plane. Inserting this data into (5.20), we conclude that

$$c_2(V^1) - \frac{1}{2}c_2(TX) = -4c_1(dP_8)\sigma - 16F \,. \tag{5.26}$$

Furthermore, let us take

$$\lambda^{(2)} = 1\,, \qquad \eta^{(2)} = 7c_1(dP_8)\,. \tag{5.27}$$

We see that this value of $\lambda^{(2)}$ satisfies (5.14) and that $\eta^{(2)} = c_1(dP_8) + 6c_1(dP_8)$ and, hence, satifies (5.15). Clearly $\eta^{(2)}$ is effective since $c_1(dP_8)$ is. Substituting this data into (5.21), (5.22) and (5.23) then gives

$$c_2(V^2) - \frac{1}{2}c_2(TX) = c_1(dP_8)\sigma + 15F \tag{5.28}$$

and

$$W = 3c_1(dP_8)\sigma + F\,, \tag{5.29}$$

respectively. Note that W satisfies (4.14) and, hence, is an effective class.

The tensions on the visible 1-plane, the hidden 2-plane and the bulk five-brane are found by inserting these expressions in (5.6), (5.12) and (5.18) respectively. To evaluate the signs of these tensions, it is necessary to recall that the integral of any effective four-form over the Poincaré dual of a Kähler class is simply the volume of the complex curve associated with that four-form. Hence, this integral is always positive. Specifically, it follows that

$$\int_{\mathcal{C}_\omega} c_1(dP_8)\sigma > 0\,, \qquad \int_{\mathcal{C}_\omega} F > 0 \tag{5.30}$$

since both $c_1(dP_8)\sigma$ and F are effective classes. Inserting eq. (5.26) into (5.6), we find

$$\alpha^{(1)} = -\frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}(4c_1(dP_8)\sigma + 16F)\,. \tag{5.31}$$

Using (5.30), we conclude that the tension on the visible 1-plane is negative. Similarly, substituting eq. (5.28) into (5.12) yields

$$\alpha^{(2)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}(c_1(dP_8)\sigma + 15F)\,. \tag{5.32}$$

Hence, the tension on the hidden 2-plane is positive. Finally, putting eq. (5.21) into (5.18) gives

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}(3c_1(dP_8)\sigma + F)\,. \tag{5.33}$$

Thus the five-brane has positive tension. As a final check on this calculation, note that

$$\alpha^{(1)} + \alpha^{(2)} + \beta = 0 \tag{5.34}$$

as it must to satisfy the topological anomaly cancellation condition.

We conclude that this vacuum, based on an elliptically fibered Calabi-Yau threefold with $B = dP_8$, has a three family, SU(5) GUT theory supported on the 1-plane.

This visible sector has $\alpha^{(1)} < 0$ and, hence, negative brane tension. On the other hand, the 2-plane supports an E_7 hidden sector which, since $\alpha^{(2)} > 0$, has positive brane tension. Anomaly cancellation requires that there be physical five-branes in the bulk, which, since they correspond to an effective class, have positive tension.

Example 2: F_r. In this example, we will choose the base surface of the elliptically fibered Calabi-Yau three-fold to be

$$B = F_r\,. \tag{5.35}$$

Using (4.4), (4.5) and the fact that $c_1(F_r)^2 = 8$, it follows that, for arbitrary r, eq. (5.7) becomes

$$c_2(V^1) - \frac{1}{2}c_2(TX) = (\eta^{(1)} - 6c_1(B))\sigma - \frac{1}{2}\left(172 - \frac{15}{\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right)\right)F\,, \tag{5.36}$$

eq. (5.13) simplifies to

$$\begin{aligned} c_2(V^2) - \frac{1}{2}c_2(TX) = {} & (\eta^{(2)} - 6c_1(B))\sigma - \\ & -\frac{1}{2}\left(96 - 2\left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}\left(\eta^{(2)} - 2c_1(B)\right)\right)F\,, \end{aligned} \tag{5.37}$$

and the five-brane class specified by (5.16) and (5.17) becomes

$$W_B = \sigma(12c_1(B) - \eta^{(1)} - \eta^{(2)}) \tag{5.38}$$

and

$$a_f = 134 - \frac{15}{2\lambda^{(1)}}\left(\lambda^{(1)2} - \frac{1}{4}\right) - \left(\lambda^{(2)2} - \frac{1}{4}\right)\eta^{(2)}\left(\eta^{(2)} - 2c_1(B)\right), \tag{5.39}$$

where

$$c_1(F_r) = 2S + (r+2)\mathcal{E}\,. \tag{5.40}$$

We now choose

$$\lambda^{(1)} = -\frac{3}{2}\,, \qquad \eta^{(1)} = 2S + (r-3)\mathcal{E}\,. \tag{5.41}$$

Note that this value of $\lambda^{(1)}$ satifies (5.8) and that $\eta^{(1)}$ is effective for any integer $r \geq 3$, which we henceforth assume. It is straightforward to show that these choices satisfy the three-family condition (5.5) on the visible plane. Inserting this data into (5.36), we conclude that

$$c_2(V^1) - \frac{1}{2}c_2(TX) = -(10S + 5(r+3)\mathcal{E})\sigma - 96F\,. \tag{5.42}$$

Furthermore, let us take

$$\lambda^{(2)} = 1\,, \qquad \eta^{(2)} = 6c_1(F_r) \tag{5.43}$$

We see that this value of $\lambda^{(2)}$ satisfies (5.14) and that $\eta^{(2)} = c_1(F_r) + 10S + 5(r+2)\mathcal{E}$ and, hence, satifies (5.15) for even integers r. We hereafter assume that r is even. Clearly $\eta^{(2)}$ is effective. Substituting this data into (5.37), (5.38) and (5.39) then gives

$$c_2(V^2) - \frac{1}{2}c_2(TX) = 96F \tag{5.44}$$

and

$$W = (10S + 5(r+3)\mathcal{E})\sigma\,, \tag{5.45}$$

respectively. Note that W satisfies (4.14) and, therefore, is an effective class.

The tensions on the visible 1-plane, the hidden 2-plane and the bulk five-brane are found by inserting these expressions in (5.6), (5.12) and (5.18) respectively. To evaluate the signs of these tensions, one notes that

$$\int_{\mathcal{C}_\omega} S\sigma > 0\,, \qquad \int_{\mathcal{C}_\omega} \mathcal{E}\sigma > 0\,, \qquad \int_{\mathcal{C}_\omega} F > 0 \tag{5.46}$$

since S, $\mathcal{E}$ and F are effective classes. Inserting eq. (5.42) into (5.6), we find

$$\alpha^{(1)} = -\frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}((10S + 5(r+3)\mathcal{E})\sigma + 96F) \tag{5.47}$$

It follows that, using (5.46), the tension on the visible 1-plane is negative for any allowed integer r. Similarly, substituting eq. (5.44) into (5.12) yields

$$\alpha^{(2)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega} 96F\,, \tag{5.48}$$

which is positive. It follows that the tension on the hidden 2-plane is positive. Finally, putting eq. (5.45) into (5.18) gives

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}(10S + 5(r+3)\mathcal{E})\sigma\,, \tag{5.49}$$

which implies that the five-brane has positive tension. As a final check on this calculation, note that

$$\alpha^{(1)} + \alpha^{(2)} + \beta = 0 \tag{5.50}$$

as it must to satisfy the topological anomaly cancellation condition.

We conclude that this vacuum, based on an elliptically fibered Calabi-Yau threefold with $B = F_r$, $r \geq 4$ and even, has a three family, SU(5) GUT theory supported on the 1-plane. This visible sector has $\alpha^{(1)} < 0$ and, hence, negative brane tension. On the other hand, the 2-plane supports an E_7 hidden sector which, since $\alpha^{(2)} > 0$, has positive brane tension. Anomaly cancellation requires that there be physical five-branes in the bulk, which, since they correspond to an effective class, have positive tension.

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These examples establish, definitively, that M-theory vacua exist in which the visible sector, hidden sector and bulk five-brane "manifestly" have negative, positive and positive tensions respectively. Furthermore, one can produce many examples of this type, indicating that such vacua are by no means exceptional. Be that as it may, such examples are only a small subset of vacua with these properties. In this larger class of vacua, the signs of the tensions are less manifest, and require further discussion.

6. Curve volume and brane tension

In this section, we greatly extend the number of vacua with the property that the tensions on the visible 1-plane, hidden 2-plane and bulk five-brane are negative, positive and positive respectively. To do this, we must first consider the relative magnitude of specific integrals over effective classes. As discussed in [19] and in the appendix, the requirement that a holomorphic vector bundle over an elliptically fibered Calabi-Yau three-fold be stable, restricts its Kähler class to a region near the boundary of the Kähler cone. This restriction is such that the volume of any effective curve in the three-fold of the form $\pi^*\eta \cdot \sigma$, that is, any "horizontal" curve, is much larger than the volume of the "vertical" fiber curve F. That is

$$\int_{\mathcal{C}_\omega} \pi^*\eta \cdot \sigma \gg \int_{\mathcal{C}_\omega} F \tag{6.1}$$

In fact, as discussed in [19] and in the appendix, the volume of any horizontal curve can be made arbitrarily large relative to the volume of the fiber curve F. This is accomplished by choosing the Kähler class sufficiently close to the Kähler cone boundary. This inequality will have important implications for determining M-theory vacua with the required tension properties. These implications are most easily demonstrated with explicit examples, to which we now turn.

Example 3: F_r. In this example, we continue to assume that equations (5.2) and (5.5) hold, leading to a three family SU(5) GUT on the visible 1-plane. We further maintain condition (5.11), which implies that the 2-plane supports a hidden sector with an E_7 gauge group. Choose the base surface of the elliptically fibered three-fold to be

$$B = F_r \, . \tag{6.2}$$

The relevant equations are then (5.36), (5.37), (5.38) and (5.39).

We now choose, as before

$$\lambda^{(1)} = -\frac{3}{2}, \qquad \eta^{(1)} = 2S + (r-3)\mathcal{E} \, . \tag{6.3}$$

This value of $\lambda^{(1)}$ satisfies (5.8) and $\eta^{(1)}$ is effective for any integer $r \geq 3$. These choices satisfy the three family condition (5.5) on the visible plane. Inserting this data into (5.36), we conclude that

$$c_2(V^1) - \frac{1}{2}c_2(TX) = -(10S + 5(r+3)\mathcal{E})\sigma - 96F\,. \tag{6.4}$$

Now, however, let us take

$$\lambda^{(2)} = 0\,, \qquad \eta^{(2)} = 22S + (11r + 26)\mathcal{E}\,. \tag{6.5}$$

We see that this value of $\lambda^{(2)}$ satisfies (5.14) and that $\eta^{(2)} = c_1(F_r) + 10c_1 + 4\mathcal{E}$ and, hence, satisfies (5.15) for any integer r. Clearly $\eta^{(2)}$ is effective. Substituting this data into (5.37), (5.38) and (5.39) then gives

$$c_2(V^2) - \frac{1}{2}c_2(TX) = (10S + (5r+14)\mathcal{E})\sigma - 286F \tag{6.6}$$

and

$$W = \mathcal{E}\sigma + 382F\,, \tag{6.7}$$

respectively. Note that W satisfies (4.14) and, therefore, is an effective class.

The tensions on the visible 1-plane, the hidden 2-plane and the bulk five-brane are found by inserting these expressions into (5.6), (5.12) and (5.18) respectively. Inserting eq. (6.4) into (5.6), we find

$$\alpha^{(1)} = -\frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}((10S + 5(r+3)\mathcal{E})\sigma + 96F)\,. \tag{6.8}$$

We conclude that, using (5.47), the tension on the visible brane is negative for any allowed integer r. Similarly, substituting eq. (6.6) into (5.12) yields

$$\alpha^{(2)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}((10S + (5r+14)\mathcal{E})\sigma - 286F)\,. \tag{6.9}$$

Here, however, we find a difference from the previous examples. This tension is not "manifestly" positive because of the minus sign in the integrand. However, it follows from (6.1) that

$$\int_{\mathcal{C}_\omega} S\sigma \gg \int_{\mathcal{C}_\omega} F\,, \qquad \int_{\mathcal{C}_\omega} \mathcal{E}\sigma \gg \int_{\mathcal{C}_\omega} F\,. \tag{6.10}$$

The Kähler class can be chosen so that, for any allowed value of r, the F term in (6.9) can be ignored relative to the S and $\mathcal{E}$ terms. For the smallest value of r, that is, $r = 3$, this can be achieved by making the volume of the each base curve greater than eight times the volume of F, which is easily done. For larger values of r, this constraint gets progressively simpler to satisfy. We conclude therefore, using (5.47), that, despite the appearance of the minus sign in the integrand, the tension on the hidden brane is positive for any allowed integer r. Finally, putting eq. (6.7) into (5.18) gives

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}(\mathcal{E}\sigma + 382F)\,, \tag{6.11}$$

which implies the five-brane has positive tension. As a final check, note that

$$\alpha^{(1)} + \alpha^{(2)} + \beta = 0 \tag{6.12}$$

as it must.

We conclude that this vacuum, based on an elliptically fibered Calabi-Yau threefold with $B = F_r$, $r \geq 3$, has a three family, SU(5) GUT supported on the 1-plane. This visible sector has $\alpha^{(1)} < 0$ and, hence, negative brane tension. On the other hand, the 2-plane supports an E_7 hidden sector. The stability of the holomorphic vector bundle V^2 is guaranteed if we choose the volume of the base curves much larger than the volume of the fiber curve F. It then follows that $\alpha^{(2)} > 0$. Hence, the hidden sector has positive brane tension. Anomaly cancellation requires that there be physical five-branes in the bulk which, since they correspond to an effective class, have positive brane tension. We remark in passing that $\alpha^{(1)}$ is a linear function of r whereas β is independent of r. This fact is physically significant, as we will discuss in the next section.

Example 4: F_r. In this example, we continue to assume that equations (5.2) and (5.5) hold, leading to a three family SU(5) GUT on the visible 1-plane. However, we now take the structure group of V^2 to be

$$G^{(2)} = \mathrm{SU}(3)\,. \tag{6.13}$$

That is, we choose

$$n^{(2)} = 3\,. \tag{6.14}$$

It follows that the low energy gauge group on the 2-plane is

$$H^{(2)} = E_6 \tag{6.15}$$

which is the commutant subgroup of $G^{(2)} = \mathrm{SU}(3)$ in E_8. We need to impose no further restrictions. Condition (6.14) implies that the 2-plane supports an E_6 supergauge theory with a certain number of matter multiplets. That is, the 2-plane is typical of a hidden sector. Note from (4.8) that, since we have chosen $n^{(2)}$ to be even, we must have

$$\lambda^{(2)} = m^{(2)} + \frac{1}{2}\,, \tag{6.16}$$

where $m^{(2)}$ is any integer. There is no further constraint on $\eta^{(2)}$.

Since the analysis is similar to that of the previous three examples, here we will simply state the results. We begin by choosing

$$\lambda^{(1)} = -\frac{3}{2}, \qquad \eta^{(1)} = 2S + (r-3)\mathcal{E}\,. \tag{6.17}$$

This value of $\lambda^{(1)}$ satisfies (5.8) and $\eta^{(1)}$ is effective for any integer $r \geq 3$. These choices satisfy the three family condition (5.5) on the visible plane. With this data, we find that

$$c_2(V^1) - \frac{1}{2}c_2(TX) = -(10S + 5(r+3)\mathcal{E})\sigma - 96F\,. \tag{6.18}$$

Now take

$$\lambda^{(2)} = \pm\frac{1}{2}, \qquad \eta^{(2)} = 22S + (11r + 27)\mathcal{E}\,. \tag{6.19}$$

This value of $\lambda^{(2)}$ satisfies (6.16) and $\eta^{(2)}$ is effective for any allowed value of r. It follows from these choices that

$$c_2(V^2) - \frac{1}{2}c_2(TX) = (10S + 5(r+3)\mathcal{E})\sigma - 54F \tag{6.20}$$

and

$$W = 150F\,. \tag{6.21}$$

Note that W satisfies (4.14) and, therefore, is an effective class.

The tensions on the visible 1-plane, the hidden 2-plane and the bulk five-brane are found by inserting these expressions into (5.6), (5.12) and (5.18) respectively. Inserting eq. (6.18) into (5.6), we find

$$\alpha^{(1)} = -\frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}((10S + 5(r+3)\mathcal{E})\sigma + 96F)\,. \tag{6.22}$$

It follows that, using (5.47), the tension on the visible brane is negative for any allowed integer r. Similarly, substituting eq. (6.20) into (5.12) yields

$$\alpha^{(2)} = \frac{2\sqrt{2}\pi}{v^{2/3}}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{C}_\omega}((10S + 5(r+3)\mathcal{E})\sigma - 54F)\,. \tag{6.23}$$

As in the previous example, this tension is not "manifestly" positive because of the minus sign in the integrand. However, it follows from (6.10) that, for any allowed value of r, the Kähler class can be chosen so that the F term in (6.23) can be ignored relative to the S and $\mathcal{E}$ terms. For the smallest value of r, that is, $r = 3$, it suffices to make the volume of each base curve greater than two times the volume of F, which is easily achieved. For larger values of r, this constraint gets progressively simpler to satisfy. We conclude therefore, using (5.47), that, despite the appearance of the minus sign in the integrand, the tension of the hidden brane is positive for any allowed integer r. Finally, putting eq. (6.21) into (5.18) gives

$$\beta = \frac{2\sqrt{2}\pi}{v^{2/3}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \int_{\mathcal{C}_\omega} 150F \tag{6.24}$$

which implies that the five-brane has positive tension. As a final check, note that

$$\alpha^{(1)} + \alpha^{(2)} + \beta = 0 \tag{6.25}$$

as it must.

We conclude that this vacuum, based on an elliptically fibered Calabi-Yau threefold with $B = F_r$, $r \geq 3$, has a three family, SU(5) GUT supported on the 1-plane. This visible sector has $\alpha^{(1)} < 0$ and, hence, negative brane tension. On the other hand, the 2-plane supports an E_6 hidden sector. Since the stability of the holomorphic vector bundle V^2 requires that we choose the volume of the base curves much larger than the volume of the fiber curve F, it follows that $\alpha^{(2)} > 0$. Hence, the hidden sector has positive brane tension. Anomaly cancellation requires that there be physical five-branes in the bulk which, since they correspond to an effective class, have positive brane tension. We remark in passing that, unlike $\alpha^{(1)}$, β depends only on the small volume fiber curve F. This fact is physically relevant, as we will discuss in the next section.

These examples establish, definitively, that there is an enormous class of M-theory vacua which have the property that the tensions on the visible sector, hidden sector and bulk five-brane are negative, positive and positive respectively.

7. The ratio $\beta/|\alpha^{(1)}|$

In the recently proposed Ekpyrotic theory of cosmology [1], a fundamental input parameter is the ratio of the five-brane tension β to the magnitude of the visible brane tension $|\alpha^{(1)}|$. In this section, we examine the values of that ratio in each of the representative examples presented above.

Example 1. Choosing the Kähler class so that

$$\int_{\mathcal{C}_\omega} c_1(dP_8)\sigma \gg \int_{\mathcal{C}_\omega} F \tag{7.1}$$

allows one to safely ignore the contribution of the fiber curve to the tensions of both the bulk five-brane and the visible 1-plane. It then follows from eqs. (5.31) and (5.33) that

$$\frac{\beta}{|\alpha^{(1)}|} \sim \frac{3}{4}\,. \tag{7.2}$$

Example 2. Choosing the Kähler class so that inequalities (6.10) are satisfied, allows one to disregard the contribution of the fiber curve to the tensions. Then, we find from equations (5.47) and (5.49) that

$$\frac{\beta}{|\alpha^{(1)}|} \sim 1\,. \tag{7.3}$$

Note that the $\beta/|\alpha^{(1)}|$ ratio in both of these examples is near unity. It follows that neither of these vacua would be suitable candidates for the Ekpyrotic cosmological scenario, at least not as presently formulated in [1]. However, as we now show, this is only true of a restricted set of M-theory vacua.

Example 3. Choosing the Kähler class so that inequalities (6.10) are satisfied, again allows one to safely ignore the contribution of the fiber curve to the tensions. Then, from equations (6.8) and (6.11) we find that

$$\frac{\beta}{|\alpha^{(1)}|} < \frac{1}{5(r+3)}\,. \tag{7.4}$$

It is important to note that the value of integer r for Hirzebruch surfaces F_r is unrestricted, and can be made as large as desired. It follows that, for large integer r, the $\beta/|\alpha^{(1)}|$ ratio can be made arbitrarily small. Such vacua are perfectly suited to support the Ekpyrotic cosmology presented in [1].

Example 4. This example differs somewhat from the previous three in that β depends only on the volume of the fiber curve. It follows from equations (6.10), (6.22) and (6.24) that

$$\frac{\beta}{|\alpha^{(1)}|} < \left(\frac{30}{r+3}\right)\frac{\int_{\mathcal{C}_\omega} F}{\int_{\mathcal{C}_\omega} \mathcal{E}\sigma}\,. \tag{7.5}$$

For any fixed value of integer r, the $\beta/|\alpha^{(1)}|$ ratio can be made as small as desired by appropriate choice of the Kähler structure. Clearly, in this type of vacua, the $\beta/|\alpha^{(1)}|$ ratio is naturally very small. Such vacua have all the properties required by the Ekpyrotic cosmological scenario.

8. Conclusion

We have shown that there are classes of BPS vacua in heterotic M-theory that exhibit negative tension on the visible orbifold plane, positive tension on the hidden plane and positive tension on the five-branes in the bulk. We chose as background geometries elliptically fibered Calabi-Yau three-folds with certain del Pezzo and Hirzebruch bases. However, it is straightforward to extend our results to any elliptically fibered Calabi-Yau three-fold. Furthermore, in our examples, we chose the low energy gauge group on the visible brane to be $G^{(1)} = \mathrm{SU}(5)$. Similar conclusions will be reached, however, if one takes other physically relevant gauge groups, such as $G^{(1)} = E_6$, SO(10) or the standard model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. In the same way, although we chose the gauge group on the hidden brane to be $G^{(2)} = E_7$ and E_6, many other choices, such as of $G^{(2)} = \mathrm{SO}(10)$, will lead to similar results. That is, the class of BPS solutions in M-theory with negative tension on the visible brane and positive tensions on the hidden brane and bulk five-branes is large and robust. Although it is hard to quantify, in practice it is as easy to find M-theory vacua with this property as to find vacua with the reverse property, that is, positive tension on the visible brane.

In addition, we examined the ratio, $\beta/|\alpha^{(1)}|$, of the tension of the bulk brane to the visible brane tension in vacua of this type. We found that, for many vacua, this ratio is of order unity. However, it was easy to construct examples in which this ratio is small. In fact, there are several different classes of physically sensible vacua for which this ratio is arbitrarily small.

We conclude that the properties of vacua required in the examples of Ekpyrotic cosmology presented in [1], that is, a negative tension visible brane, a positive tension hidden brane, positive tension bulk five-branes and a small $\beta/|\alpha^{(1)}|$ ratio, are found among large classes of physically realistic vacua of heterotic M-theory.

Acknowledgments

We would like to thank Andre Lukas and Dan Waldram for helpful conversations. This work was supported in part by the National Science Foundation grant DMS-9802456 (R.Y.D.), the Natural Sciences and Engineering Research Council of Canada (J.K.), the US Department of Energy grants DE-FG02-91ER40671 (J.K. and P.J.S.), DE-FG02-96ER40959 and DE-AC02-76-03071 (B.A.O.), and by PPARC-UK (N.T.).

A. Stability of vector bundles

In order to preserve $N = 1$ supersymmetry, the vector bundles V^n must be stable. (Actually, it suffices for them to be poly-stable. Poly-stability is a property intermediate between stability and semi-stability. All the bundles we construct in this paper are actually stable, so we do not need to worry about poly-stability here.)

It is important to note that the notion of stability of a vector bundle V depends on the choice of the Kähler class ω on the elliptically fibered Calabi-Yau three-fold X. The set of all possible Kähler classes forms the Kähler cone. This is an open, convex cone in the vector space $H^2(X, \mathbb{R})$, given by the condition that the integral of ω^i on each effective i-dimensional complex subspace of X is positive. In particular, the integral of ω itself on each effective curve must be positive. This fact was used in equations (5.30) and (5.46) in the text.

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It is known that as ω varies in the Kähler cone, the moduli space of stable bundles can jump around as a result of bundles which are stable with respect to one Kähler class becoming unstable with respect to another. Typically, the Kähler cone is divided by walls into open chambers; the moduli space is independent of ω in the interior of each chamber, but undergoes a flop-like transition when a wall is crossed.

Now the spectral construction [13, 14] produces bundles V on X with the property that the restriction $V|_F$ to a general elliptic fiber F is semi-stable. (But not stable: typically $V|_F$ is either a sum of several line bundles, all of the same slope, or an extension involving such line bundles.) Let w be a Kähler class on the base B. Then w^2 is some positive multiple of the class of a point in B. Therefore, for the pullback π^*w on X, we have that $(\pi^*w)^2$ is some positive multiple of the class of the fiber F. The semi-stability of $V|_F$, therefore, amounts to saying that V is "semi-stable with respect to π^*w".

This result is incomplete on two counts. First, we need to strengthen semi-stability to stability. But more importantly, π^*w is not a Kähler class on X. Its integral on each effective curve C in X is indeed non-negative: it is the same as the integral of the original Kähler class w on the image $\pi(C)$ in B. However, there is one curve, namely the fiber F, for which the image $\pi(C)$ and, hence, also the integral, vanish. This means that π^*w is on the boundary of the Kähler cone. We must move to the interior in order for stability of V to make sense.

It turns out that we can indeed cure both problems simultaneously by moving ever so slightly into the interior of the Kähler cone, specifically into the chamber which is closest to the given boundary point π^*w. To do this, choose any Kähler class ω_0 on X, and consider the combination

$$\omega = \pi^*w + \epsilon\omega_0 \,, \qquad \epsilon > 0 \,.$$

ω, unlike π^*w, is a Kähler class on X. It has the property that for any effective class c in $H^2(B,\mathbb{Z})$, the volume of $\sigma \cdot \pi^*c$ with respect to ω on X is at least the volume of c with respect to the Kähler class w on the base B. On the other hand, the volume of F with respect to ω is $\epsilon \int_{\mathcal{C}_{\omega_0}} F$. By taking ϵ sufficiently small, we can thus make the volume of F arbitrarily small compared to the volume of effective classes associated with the base. This result was used in equations (6.1), (6.10) and (7.1) in the text.

The crucial property of this Kähler class ω is that, for ϵ sufficiently small, a vector bundle V which arises by the spectral construction from an irreducible spectral cover is actually stable with respect to this choice of ω. This was proven in theorem 7.1 of [15]. It has been used in many of the heterotic M-theory constructions including [18], where it was employed to reduce the GUT gauge group SU(5) to the standard model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. (The stability of V is discussed in section 5.2 of that reference, and the specific Kähler class ω used there is given in (6.1.3).) In the present situation, this implies that the bundle V can be taken to be stable with respect to a Kähler class ω while keeping the volume of the fiber F arbitrarily small compared to the volumes of effective curves associated with the base.

Let us explain why V is stable with respect to ω. Stability means that the slope $\mu_\omega(W)$ of any proper sub-bundle (or sub-sheaf) $W \subset V$ is strictly less than $\mu_\omega(V)$. Here the slope (with respect to ω) is defined as $\mu_\omega(W) = \frac{c_1(W)\cdot\omega^2}{\mathrm{rank}(W)}$. Even though $\pi^* w$ is in the boundary of the Kähler cone, and so is not a Kähler class on X, we can still define the slope $\mu_{\pi^* w}(W) = \frac{c_1(W)\cdot(\pi^* w)^2}{\mathrm{rank}(W)}$ with respect to it. Since $(\pi^* w)^2$ is some positive multiple of the class of the fiber F, the resulting notion of "semi-stability with respect to $\pi^* w$" is implied by semi-stability of the restrictions $V|_F$ to the fibers. Assume that V is not stable with respect to ω, then there is a "destabilizing" sub-bundle $W \subset V$ with $\mu_\omega(W) \geq \mu_\omega(V)$. But semi-stability along the fibers says that $\mu_{\pi^* w}(W) \leq \mu_{\pi^* w}(V)$. If we had equality, it would follow that W arises by the spectral construction from a proper sub-variety of the spectral cover of V, contradicting the assumption that this cover is irreducible. So we must have a strict inequality $\mu_{\pi^* w}(W) < \mu_{\pi^* w}(V)$. But then it is easy to see that, by taking ϵ small enough, we can also ensure that $\mu_\omega(W) < \mu_\omega(V)$, so W cannot destabilize V after all. (For more details, see [15].)

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